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EDGE SINGULARITIES IN ANISOTROPIC COMPOSITES

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Abstract—The stress singularity at the vertex of an anistropic wedge has the form $r^{-\xi}F(r,\theta)$ as $r \to 0$ where $0 < \xi < 1$ and F is a real function of the polar coordinates (r, θ). In many cases, F is independent of r. The explicit form of $F(r,\theta)$ depends on the eigenvalues of the elasticity constants, called p here and on the order of singularity κ. When κ is real, $\xi = \kappa$. If κ is complex, ξ is the real part of κ. The p's are all complex and consist of 3 pairs of complex conjugates which reduce to $\pm i$ when the material is isotropic. The function F depends not only on p and κ, it also depends on whether p and κ are distinct roots of the corresponding determinants. In this paper we present the function $F(r,\theta)$ in terms of p and κ for the cases when p and κ are single roots as well as when they are multiple roots. The relationship between the complex variable Z introduced in the analysis and the polar coordinates (r,θ) is interpreted geometrically. After presenting the form of F for individual cases, a general form of F is given in eqn (74). We also show that the stress singularity at the crack tip of general anisotropic materials has the order of singularity $\xi = 1/2$ which is at least a multiple root of order 3. The implication of this on the form $F(r,\theta)$ is discussed.

1. INTRODUCTION

For isotropic materials, use of the biharmonic function, or the Airy stress function, seems to be the universal approach in the analysis of stress singularities[1–4]. There appears to be no universal approach in analyzing the stress singularities in anisotropic materials. Lekhnitskii[5] introduced two stress functions to analyze general anisotropic materials. His approach was used by Wang and Choi[6] to study the thermal stresses at the interface in a layered composite. Green and Zerna[7] employed a complex function representation of the solution. Their approach was used by Bogy[8] and Kuo and Bogy[9] in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. It should be mentioned that plane deformation was assumed in[7–9] and hence the material property was assumed to be symmetric with respect to the plane of deformation.

In this paper we use the approach of Stroh[10] whose analysis was further developed by Barnett *et al.* (see, e.g. [11]) to study the surface waves in anisotropic elastic materials. An excellent review article on surface waves in anisotropic elastic materials was given by Chadwick and Smith[12]. Although no stress singularities were studied in[10–12], their approach is used here to find the stress distribution at an anisotropic wedge. A recent study by Dempsey and Sinclair[3] on isotropic elastic wedge problems shows that the singularity analysis can be accomplished without resorting to the Mellin transform even when the boundary conditions are not homogeneous[4]. Following their analysis and using the approach of Stroh, we present here possible forms of stress distribution near the vertex of a wedge or a composite wedge of anisotropic materials.

The stress distribution near the vertex of a wedge or a composite wedge depends on whether the eigenvalues p of the elasticity constants are distinct. It also depends on whether the order of singularity κ is a single or multiple root. The purpose of this paper is to show how one can derive the form of stress distribution when p and/or κ are not single roots. We also show the geometrical meaning of the complex variable Z in terms of the polar coordinates (r,θ).

Finally, as an application, we consider the stress singularity at a crack tip of general anisotropic materials.

2. BASIC EQUATIONS

In a fixed rectangular coordinates x_i, $(i = 1, 2, 3)$, let u_i, σ_{ij} and ϵ_{ij} be the displacement, stress and strain, respectively. The continuity condition, the stress-strain law and the equations of equilibrium can be written as

$$\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2 \tag{1}$$

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{2}$$

$$\partial\sigma_{ij}/\partial x_j = 0 \tag{3}$$

where

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} \tag{4}$$

are the elasticity constants of the anisotropic material. Unless stated otherwise, repeated indices imply summation.

We assume that u_i and σ_{ij} are independent of the x_3-coordinate. Hence we assume that

$$Z = x_1 + px_2 \tag{5}$$

$$u_i = \upsilon_i f(Z) \tag{6}$$

$$\sigma_{ij} = \tau_{ij}\,\mathrm{d}f(Z)/\mathrm{d}Z \tag{7}$$

where f is an arbitrary function of Z and p is an eigenvalue of the elasticity constants to be determined shortly. υ_i and τ_{ij} are independent of x_1 and x_2 but they depend on the eigenvalue p. Substitution of eqns (6) and (7) into eqns (1)–(3) yields the results

$$\tau_{ij} = (c_{ijk1} + pc_{ijk2})\upsilon_k \tag{8}$$

$$D_{ik}\upsilon_k = 0 \tag{9}$$

where

$$D_{ik} = c_{i1k1} + p(c_{i1k2} + c_{i2k1}) + p^2 c_{i2k2}. \tag{10}$$

For a non-trivial solution of υ_k, the determinant of D_{ik} must vanish. This provides the eigenvalues p. Equation (9) then provides the eigenvector υ_i.

3. EIGENVALUES AND EIGENVECTORS OF THE ELASTICITY CONSTANTS

In view of eqn (4), c_{ijkl} has only 21 constants. If we write eqns (2) and (4) as

$$\sigma_i = c_{ij}\epsilon_j, \quad c_{ij} = c_{ji} \tag{11}$$

where

$$\left.\begin{array}{lll} \sigma_1 = \sigma_{11}, & \sigma_2 = \sigma_{22}, & \sigma_3 = \sigma_{33}, \\ \sigma_4 = \sigma_{23}, & \sigma_5 = \sigma_{13}, & \sigma_6 = \sigma_{12}, \end{array}\right\} \tag{12a}$$

$$\left.\begin{array}{lll} \epsilon_1 = \epsilon_{11}, & \epsilon_2 = \epsilon_{22}, & \epsilon_3 = \epsilon_{33}, \\ \epsilon_4 = 2\epsilon_{23}, & \epsilon_5 = 2\epsilon_{13}, & \epsilon_6 = 2\epsilon_{12}. \end{array}\right\} \tag{12b}$$

the coefficients in eqn (10) can be written as

$$\begin{aligned} Q_{ik} &= c_{i1k1} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{61} & c_{66} & c_{65} \\ c_{51} & c_{56} & c_{55} \end{bmatrix} \\ R_{ik} &= c_{i1k2} = \begin{bmatrix} c_{16} & c_{12} & c_{14} \\ c_{66} & c_{62} & c_{64} \\ c_{56} & c_{52} & c_{54} \end{bmatrix} \\ T_{ik} &= c_{i2k2} = \begin{bmatrix} c_{66} & c_{62} & c_{64} \\ c_{26} & c_{22} & c_{24} \\ c_{46} & c_{42} & c_{44} \end{bmatrix}. \end{aligned} \tag{13}$$

Equation (10) can then be written as

$$D_{ik} = Q_{ik} + p(R_{ik} + R_{ki}) + p^2 T_{ik} \tag{14}$$

and vanishing of the determinant D_{ik} means

$$\begin{vmatrix} c_{11}+2pc_{16}+p^2c_{66} & c_{16}+p(c_{12}+c_{66})+p^2c_{26} & c_{15}+p(c_{14}+c_{56})+p^2c_{46} \\ c_{16}+p(c_{12}+c_{66})+p^2c_{26} & c_{66}+2pc_{26}+p^2c_{22} & c_{56}+p(c_{25}+c_{46})+p^2c_{24} \\ c_{15}+p(c_{14}+c_{56})+p^2c_{46} & c_{56}+p(c_{25}+c_{46})+p^2c_{24} & c_{55}+2pc_{45}+p^2c_{44} \end{vmatrix} = 0. \tag{15}$$

Equation (15) provides six eigenvalues of p.

For each of p the associated v_i's are obtained from eqn (9). In general, v_i, $(i = 1,2,3)$ are all non-zero. Hence, u_1, u_2 and u_3 are coupled.

As to τ_{ij}, we let $j = 1$ and 2, respectively, in eqn (8) and use the notations of eqn (13) to obtain

$$\left.\begin{aligned} \tau_{i1} &= (Q_{ik} + pR_{ik})v_k \\ t_{i2} &= (R_{ki} + pT_{ik})v_k \end{aligned}\right\}. \tag{16a}$$

It follows from eqns (9), (14) and (16a) that

$$\tau_{i1} + p\tau_{i2} = 0 \tag{16b}$$

and hence

$$\tau_{12} = -p\tau_{22}, \quad \tau_{11} = p^2\tau_{22}, \quad \tau_{13} = -p\tau_{23}. \tag{16c}$$

Therefore, of the six components τ_{ij}, all we need is τ_{22}, τ_{33} and τ_{23}. They are obtained from eqn (8) which can be casted in the following form:

$$\tau_i = (c_{i1} + pc_{i6})v_1 + (c_{i6} + pc_{i2})v_2 + (c_{i5} + pc_{i4})v_3, \tag{17}$$

where τ_{ij} have been written as τ_i using the same rules defined in eqn (12a) for σ_{ij}. Equation (17) is valid for $i = 1$–6 although all we need is τ_2, τ_3 and τ_4.

Notice that since Q_{ik} and T_{ik} are symmetric, so is D_{ik}. Notice also that c_{3j}, $(j = 1,2,\ldots,6)$ are not present in eqn (15). Therefore, the eigenvalues p are independent of these elastic constants. In fact, the stress singularities are also independent of these elastic constants.

Equation (15) is a sextic equation in p. If the strain energy is positive definite, it can be shown that p cannot be real[10, 12]. Therefore, we would have 3 pairs of complex conjugate roots for p.

When the material property is symmetric with respect to the (x_2, x_3) plane or to the (x_1, x_3) plane, eqn (15) reduces to a cubic in p^2[10]. Since every cubic has at least one real root, one of the p's will be purely imaginary when (x_2, x_3) or (x_1, x_3) is a plane of symmetry.

4. UNCOUPLING OF u_3 FROM u_1 AND u_2

When the material property is symmetric with respect to the (x_1, x_2) plane, we have

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0. \tag{18}$$

Equation (15) then reduces to

$$\begin{vmatrix} c_{11}+2pc_{16}+p^2c_{66} & c_{16}+p(c_{12}+c_{66})+p^2c_{26} & 0 \\ c_{16}+p(c_{12}+c_{66})+p^2c_{26} & c_{66}+2pc_{26}+p^2c_{22} & 0 \\ 0 & 0 & c_{55}+2pc_{45}+p^2c_{44} \end{vmatrix} = 0. \tag{19}$$

Therefore, instead of a sextic we have a quartic equation and a quadratic equation in p.

If p is a root of the quartic equation, we see from eqns (9) and (19) that $v_3 = 0$. Moreover, eqn (17) shows that $\tau_{13} = \tau_{23} = 0$. Hence, we have a plane deformation.

Similarly, if P is a root of the quadratic equation, $v_1 = v_2 = 0$ and $\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$. This is an anti-plane deformation.

Therefore, when eqn (18) holds, the plane deformation and the anti-plane deformation are uncoupled. Since the system is linear, we may consider them separately when eqn (18) holds.

As an example, consider isotropic materials in which the only nonzero c_{ij} are

$$\left.\begin{aligned} c_{44} = c_{55} = c_{66} = \mu \\ c_{12} = c_{21} = c_{13} = c_{31} = c_{23} = c_{32} = \lambda \\ c_{11} = c_{22} = c_{33} = \lambda + 2\mu \end{aligned}\right\} \tag{20}$$

where λ and μ are the Lame constants. Equation (19) reduces to

$$\mu^2(\lambda + 2\mu)(p^2 + 1)^3 = 0 \tag{21}$$

and hence $p = \pm i$ is a triple root. However, since u_3 is uncoupled from u_1 and u_2, $p = \pm i$ is a double root for plane deformation and a single root for anti-plane deformation. The eigenvectors v_i and τ_{ij} are obtained from eqns (9) and (17). We have, for $p = i$,

$$v_i = \frac{1}{2\mu}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \tau_{ij} = \begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22a}$$

for plane deformation and

$$v_i = \frac{1}{\mu}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tau_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{bmatrix} \tag{22b}$$

for anti-plane deformation.

5. GEOMETRICAL INTERPRETATION OF $Z = x_1 + px_2$

Let α and β be, respectively, the real and imaginary part of p so that

$$p = \alpha + \beta i, \quad \beta > 0. \tag{23}$$

We assumed $\beta > 0$ because the conjugate of p will have the negative imaginary part. Using the polar coordinates with the origin at $x_1 = x_2 = 0$, we have

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta. \tag{24}$$

Hence,

$$Z = x_1 + px_2 = X + iY = r\rho\, e^{i\psi} \tag{25}$$

where

$$\left.\begin{aligned} X/r = \cos\theta + \alpha\sin\theta = \rho\cos\psi \\ Y/r = \beta\sin\theta = \rho\sin\psi \\ \rho^2 = (\cos\theta + \alpha\sin\theta)^2 + \beta^2\sin^2\theta \end{aligned}\right\}. \tag{26}$$

It is not difficult to show from eqns (24)–(26) that a unit circle in the (x_1, x_2) plane maps an ellipse in the (X, Y) plane, Fig. 1. If the (x_1, x_2) plane is a stretchable sheet, one obtains the

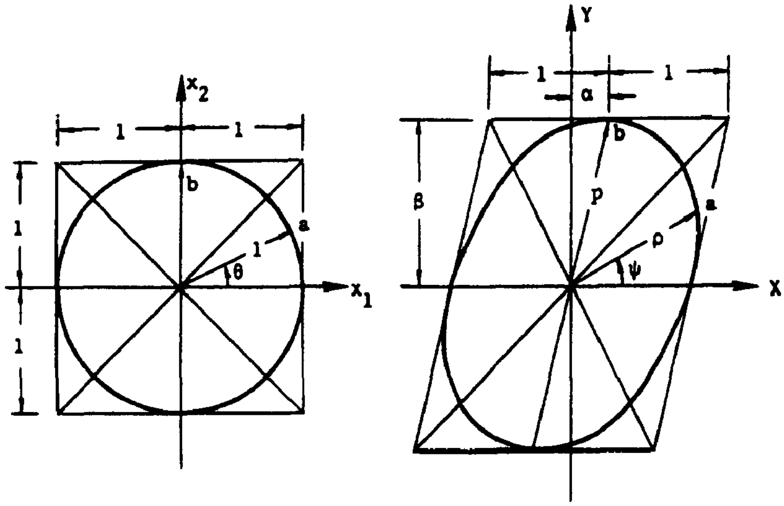

Fig. 1. Geometrical interpretation of $Z = x_1 + px_2 = r\rho\, e^{i\psi}$.

ellipse by first stretching the circle uniformly β units in the x_2-direction and then shear the sheet with the x_1-axis fixed until point b displaces α unit horizontally. From point a in (x_1, x_2) and (X, Y) planes we see the geometrical relationship between θ, ρ and ψ. From eqn (26), notice that ρ and ψ depend on θ and p but are independent of r. Notice also that

$$\rho = 1, \quad \psi = \theta, \quad \text{at } \theta = 0, \quad \pm\pi. \tag{27a}$$

If p is purely imaginary, we also have, in addition to eqn (27a),

$$\rho = \beta, \psi = \theta, \quad \text{at } \theta = \pm\pi/2, \pm 3\pi/2 \quad \text{when } \alpha = 0. \tag{27b}$$

For isotropic materials, $p = \pm i$ is a multiple root of order 3. Thus the ellipse in the (X,Y) plane reduces to a unit circle. Hence,

$$\rho = 1, \quad \psi = \theta, \tag{28}$$

and

$$Z = x_1 + ix_2 = r\, e^{i\theta} \tag{29}$$

which is the well-known complex coordinate for (x_1, x_2) in two-dimensional elasticity problems of isotropic materials.

6. STRESS DISTRIBUTION NEAR THE VERTEX WHEN p's ARE DISTINCT

To find the stress distribution and the stress singularities at the vertex of a wedge, we choose

$$f(Z) = \frac{1}{1-\kappa} Z^{1-\kappa} \tag{30}$$

where κ is the order of singularity to be determined by the boundary conditions. As we mentioned earlier, the eigenvalues p are all complex numbers and consist of three pairs of complex conjugates. In this section we assume that the eigenvalues are distinct. Using eqn (30) in eqns (6) and (7) for all eigenvalues and forming a linear combination of them leads to

$$u_i = (A_1 v_i Z^{1-\kappa} + B_1 \bar{v}_i \bar{Z}^{1-\kappa})/(1-\kappa) + \cdots \tag{31}$$

$$\sigma_{ij} = A_1 \tau_{ij} Z^{-\kappa} + B_1 \bar{\tau}_{ij} \bar{Z}^{-\kappa} + \cdots \tag{32}$$

where A_1, B_1,... are constants which may be complex and an overbar denotes a complex conjugate. For simplicity only the terms associated with one pair of eigenvalues are written explicitly to avoid introducing an additional subscript for the eigenvalues. The dots denote terms associated with the remaining two pairs of eigenvalues.

It should be pointed out that υ_k as given by eqn (9) is not unique and can have an arbitrary multiplicative constant. The constants A_1 and B_1 in eqns (31) and (32) represent this arbitrary multiplicative constant.

For a wedge or a composite wedge, by substituting eqns (31) and (32) in the boundary conditions (which include the interface conditions if the wedge is a composite), one obtains a system of linear algebraic equations in $A_1, B_1, \ldots$, which may be written as

$$K_{ij}c_j = q_i \tag{33}$$

where K_{ij} is a square matrix which depends on κ, c_j is a column matrix whose elements are A_1, $B_1, \ldots$, and q_i is a column matrix which depends on the boundary conditions. If the boundary conditions are homogeneous, $q_i = 0$. In this case, a nontrivial solution exists if the determinant of K_{ij} vanishes. The roots of this determinant provides the values for κ. Let

$$\kappa + \xi + \eta i \tag{34a}$$

where ξ and η are real. If $0 < \xi < 1$, we have a singularity at $r = 0$.

Since u_i and σ_{ij} are real, only the real parts or the imaginary parts on the right-hand sides of eqns (31) and (32) should be considered. They will have different expressions depending on if the root κ is real or complex. To this end, we write υ_i and τ_{ij} as

$$\upsilon_i = v_i \, e^{ia_i}, \quad \tau_{ij} = t_{ij} \, e^{ib_{ij}} \tag{34b}$$

where v_i, a_i, t_{ij} and b_{ij} are real and repeated indices do not imply summation here.

(a) $\kappa = \xi$, *real*

It follows from eqn (25) that the real parts of eqns (31) and (32) can be written as

$$u_i = (r\rho)^{1-\xi} v_i \{M_1 \cos[a_i + (1-\xi)\psi] + N_1 \sin[a_i + (1-\xi)\psi]\}/(1-\xi) + \cdots \tag{35}$$

$$\sigma_{ij} = (r\rho)^{-\xi} t_{ij} \{M_1 \cos(b_{ij} - \xi\psi) + N_1 \sin(b_{ij} - \xi\psi)\} + \cdots \tag{36}$$

where $M_1, N_1, \ldots$ are related to $A_1, B_1, \ldots$ and are real. The imaginary parts of eqns (31) and (32) provide no new expressions.

(b) $\kappa = \xi + i\eta$, *complex*

When κ is a complex root there is no loss in generality in assuming $\eta > 0$ because if κ is a root, so is $\bar{\kappa}$. We then have

$$\begin{aligned} Z^{-\kappa} &= (r\rho \, e^{i\psi})^{-\xi - i\eta} = (r\rho)^{-\xi} e^{\eta\psi} e^{-i\xi\psi} (r\rho)^{-i\eta} \\ &= (r\rho)^{-\xi} e^{\eta\psi} e^{-i(\xi\psi + \eta \ln(r\rho))}. \end{aligned} \tag{37}$$

The real parts, or the imaginary parts of eqn (32) now become

$$\sigma_{ij} = (r\rho)^{-\xi} t_{ij} \{e^{\eta\psi}(M_1^- \cos\phi_{ij}^- + N_1^- \sin\phi_{ij}^-) + e^{-\eta\psi}(M_1^+ \cos\phi_{ij}^+ + N_1^+ \sin\phi_{ij}^+)\} + \cdots \tag{38}$$

where

$$\phi_{ij}^{\pm} = b_{ij} - \xi\psi \pm \eta \ln(r\rho) \tag{39}$$

and $M_1^{\pm}$, $N_1^{\pm}$ are real constants and are related to A_1 and B_1. A similar equation may be written for u_i. We see that σ_{ij} is oscilatory and unbounded as $r \to 0$. As expected, eqn (38) reduces to eqn (36) when $\eta = 0$.

In the sequel, we will consider only the cases in which κ is a complex. The solution for a real κ is deduced by letting $\eta = 0$.

7. STRESS DISTRIBUTION NEAR THE VERTEX WHEN p IS A DOUBLE ROOT

When p is a double root of eqn (15), we have only two pairs of distinct eigenvalues instead of three. It is not difficult to see that, if

$$u = v_i Z^{1-\kappa}/(1-\kappa) \tag{40}$$

$$\sigma_{ij} = \tau_{ij} Z^{-\kappa} \tag{41}$$

are the solutions corresponding to the double root p, so are

$$\left.\begin{aligned} u_i &= \frac{1}{1-\kappa}\frac{\mathrm{d}}{\mathrm{d}p}\{v_i Z^{1-\kappa}\} \\ &= \frac{1}{1-\kappa} v_i' Z^{1-\kappa} + v_i Z^{-\kappa} x_2 \end{aligned}\right\} \tag{42}$$

$$\sigma_{ij} = \tau_{ij}' Z^{-\kappa} - \kappa \tau_{ij} Z^{-\kappa-1} x_2 \tag{43}$$

where a prime stands for differentiation with respect to p. Since

$$x_2 = (Z-\bar{Z})/(p-\bar{p}) = (Z-\bar{Z})/(2\beta i) \tag{44}$$

we have

$$u_i = \left(\frac{1}{1-\kappa} v_i' + \frac{1}{2\beta i} v_i\right) Z^{1-\kappa} - \frac{1}{2\beta i} v_i \bar{Z} Z^{-\kappa} \tag{45}$$

$$\sigma_{ij} = \left(\tau_{ij}' - \frac{\kappa}{2\beta i}\tau_{ij}\right) Z^{-\kappa} + \frac{\kappa}{2\beta i}\tau_{ij} \bar{Z} Z^{-\kappa-1} \tag{46}$$

v_i' and τ_{ij}' are obtained by differentiating eqns (9) and (8) with respect to p:

$$D_{ik} v_k' + D_{ik}' v_1 = 0 \tag{47a}$$

$$\tau_{ij}' = (c_{ijk1} + p c_{ijk2}) v_k' + c_{ijk2} v_k. \tag{47b}$$

The existence of a solution for v_k and v_k' from eqns (9) and (47a) will not be discussed here.

Since a linear combination of two independent solutions is also an independent solution, we will linearly combine eqns (40) and (41) with eqns (45) and (46), respectively, such that the term $\tau_{ij} Z^{-\kappa}$ is eliminated. Therefore, the second pair of the solution in eqns (31) and (32) when p is a double root may be written as

$$\begin{aligned} u_i = {} & A_2\{(v_i + 2\beta i v_i') Z^{1-\kappa}/(1-\kappa) - v_i \bar{Z} Z^{-\kappa}\} \\ & + B_2\{(\bar{v}_i - 2\beta i \bar{v}_i') \bar{Z}^{1-\kappa}/(1-\kappa) - \bar{v}_i Z \bar{Z}^{-\kappa}\} \end{aligned} \tag{48}$$

$$\begin{aligned} \sigma_{ij} = {} & A_2\{2\beta i \tau_{ij}' Z^{-\kappa} + \kappa \tau_{ij} \bar{Z} Z^{-\kappa-1}\} \\ & + B_2\{-2\beta i \bar{\tau}_{ij}' \bar{Z}^{-\kappa} + \kappa \bar{\tau}_{ij} Z \bar{Z}^{-\kappa-1}\} \end{aligned} \tag{49}$$

where A_2, B_2 are arbitrary constants.

Notice that if v_i' and τ_{ij}' are solutions of eqns (47), so are $v_i' + A^* v_i$ and $\tau_{ij}' + A^* \tau_{ij}$ where A^* is an arbitrary constant. It can be shown however that this does not generate additional arbitrary constants and the constants A_1, B_1, A_2 and B_2 introduced in eqns (31), (32), (48) and (49) are sufficient. One could exploit this non-uniqueness feature to find a simpler solution for v_i' and τ_{ij}'.

The real or imaginary parts of eqn (49) can be written down using eqn (37). The $Z^{-\kappa}$ and $\bar{Z}^{-\kappa}$ terms are similar to eqn (32) and hence would yield the expression given by eqn (38). The remaining terms in eqn (49) yield the following new expression:

$$\sigma_{ij} = (r\rho)^{-\xi} t_{ij}\{e^{\eta\psi}[M_2^- \cos(\phi_{ij}^- - 2\psi) + N_2^- \sin(\phi_{ij}^- - 2\psi)] + e^{-\eta\psi}[M_2^+ \cos(\phi_{ij}^+ - 2\psi) + N_2^+ \sin(\phi_{ij}^+ - 2\psi)]\} \quad (50)$$

where $\phi_{ij}^{\pm}$ are defined in eqn (39) and $M_2^{\pm}$, $N_2^{\pm}$ are related to A_2, B_2, p and κ. Equation (50) applies to the case when κ is complex. For a real κ, we simply let $\eta = 0$.

8. STRESS NEAR THE VERTEX OF AN ISOTROPIC WEDGE

For isotropic materials, $p = \pm i$ is a triple root. However, since u_3 is uncoupled from u_1 and u_2 for isotropic materials, p is actually a double root when we consider u_1 and u_2 only. Hence the previous section on a double root p applies to isotropic materials in plane deformations. We will consider plane deformation and anti-plane deformation separately.

Plane deformation

If we use eqn (22a) in eqns (47), a solution for v_i' and τ_{ij}' can be written as

$$v_i' = \frac{1}{2\mu}\begin{bmatrix}(1-2k)i \\ -2k \\ 0\end{bmatrix}, \quad \tau_{ij}' = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -2(1-k)i \end{bmatrix} \quad (51)$$

where

$$2k = \frac{\lambda + 2\mu}{\lambda + \mu}. \quad (52)$$

The solution for u_i and σ_{ij} are obtained by combining the r.h.s. of eqns (31) and (32) with (48) and (49), respectively. Introducing the new constants A, B, C, D by

$$A_1 = \bar{B}_1 = (A + iB)/2, \quad A_2 = \bar{B}_2 = (C + iD)/2 \quad (53)$$

and using eqns (29), (22a) and (51), we obtain

$$u_1 = \frac{r^{1-\kappa}}{2\mu(1-\kappa)}\{[A - C(1-4k)]\cos(1-\kappa)\theta - [B - D(1-4k)]\sin(1-\kappa)\theta - (1-\kappa)[C\cos(1+\kappa)\theta + D\sin(1+\kappa)\theta]\} \quad (54a)$$

$$u_2 = \frac{r^{1-\kappa}}{2\mu(1-\kappa)}\{-[A + C(1-4k)]\sin(1-\kappa)\theta - [B + D(1-4k)]\cos(1-\kappa)\theta - (1-\kappa)[C\sin(1+\kappa)\theta - D\cos(1+\kappa)\theta]\} \quad (54b)$$

$$\sigma_{11} = r^{-\kappa}\{(A+2C)\cos\kappa\theta + (B+2D)\sin\kappa\theta + \kappa[C\cos(2+\kappa)\theta + D\sin(2+\kappa)\theta]\} \quad (55a)$$

$$\sigma_{22} = r^{-\kappa}\{-(A-2C)\cos\kappa\theta - (B-2D)\sin\kappa\theta - \kappa[C\cos(2+\kappa)\theta + D\sin(2+\kappa)\theta]\} \quad (55b)$$

$$\sigma_{12} = r^{-\kappa}\{A\sin\kappa\theta - B\cos\kappa\theta + \kappa[C\sin(2+\kappa)\theta - D\cos(2+\kappa)\theta]\} \quad (55c)$$

$$\sigma_{33} = r^{-\kappa}4(1-\kappa)\{C\cos\kappa\theta + D\sin\kappa\theta\}. \quad (55d)$$

The other components of u_i and σ_{ij} vanish. In terms of the polar coordinates, eqns (54) and (55a–c) are transformed into the following well-known expressions[1]:

$$u_r = \frac{r^{1-\kappa}}{2\mu(1-\kappa)}\{A\cos(2-\kappa)\theta - B\sin(2-\kappa)\theta - (2-\kappa-4k)[C\cos\kappa\theta + D\sin\kappa\theta]\} \quad (56a)$$

$$u_\theta = \frac{r^{1-\kappa}}{2\mu(1-\kappa)}\{-A\sin(2-\kappa)\theta - B\cos(2-\kappa)\theta$$
$$+(\kappa-4k)[C\sin\kappa\theta - D\cos\kappa\theta]\} \quad (56b)$$
$$\sigma_{rr} = r^{-\kappa}\{A\cos(2-\kappa)\theta - B\sin(2-\kappa)\theta$$
$$+(2+\kappa)[C\cos\kappa\theta + D\sin\kappa\theta]\} \quad (57a)$$
$$\sigma_{\theta\theta} = r^{-\kappa}\{-A\cos(2-\kappa)\theta + B\sin(2-\kappa)\theta$$
$$+(2-\kappa)[C\cos\kappa\theta + D\sin\kappa\theta]\} \quad (57b)$$
$$\sigma_{r\theta} = r^{-\kappa}\{-A\sin(2-\kappa)\theta - B\cos(2-\kappa)\theta$$
$$+\kappa[C\sin\kappa\theta - D\cos\kappa\theta]\}. \quad (57c)$$

Anti-plane deformation

For anti-plane deformation, $p = \pm i$ is a single root and hence eqns (31) and (32) apply. Using eqns (22b) and (29), we have

$$u_3 = \frac{r^{1-\kappa}}{\mu(1-\kappa)}\{A\cos(1-\kappa)\theta - B\sin(1-\kappa)\theta\} \quad (58)$$
$$\sigma_{13} = r^{-\kappa}\{A\cos\kappa\theta + B\sin\kappa\theta\} \quad (59a)$$
$$\sigma_{23} = r^{-\kappa}\{A\sin\kappa\theta - B\cos\kappa\theta\}. \quad (59b)$$

In terms of the circular cylindrical coordinates, eqns (59) are replaced by the known solution[13]:

$$\sigma_{rz} = r^{-\kappa}\{A\cos(1-\kappa)\theta - B\sin(1-\kappa)\theta\} \quad (60a)$$
$$\sigma_{r\theta} = r^{-\kappa}\{-A\sin(1-\kappa)\theta - B\cos(1-\kappa)\theta\}. \quad (60b)$$

9. STRESS DISTRIBUTION NEAR THE VERTEX WHEN p IS A TRIPLE ROOT

We have not seen an example other than isotropic materials for which p is a triple root. If there is one, and if u_3 is not uncoupled from u_1 and u_2, we see that a third independent solution is

$$u_i = \frac{1}{1-\kappa}\frac{d^2}{dp^2}\{v_i Z^{1-\kappa}\} \quad (61)$$
$$\sigma_{ij} = \frac{d^2}{dp^2}\{\tau_{ij} Z^{-\kappa}\}. \quad (62)$$

Following a similar procedure in deriving eqn (50), the real expressions for the third independent solution when p is a triple root can be obtained from eqn (50) with 2ψ replaced by 4ψ and the subscripts 2 replaced by 3.

10. STRESS NEAR THE VERTEX WHEN κ IS A DOUBLE ROOT

Up to now, we tacitly assumed that κ is a single root of the determinant of K_{ij} and hence, other than a multiplicative constant, the homogeneous equation of eqn (33) has a unique solution for c_j whose elements are the coefficients $A_1, B_1, \ldots$ If κ is a multiple root, then $A_1, B_1, \ldots$ may not be unique and we have other new solutions.

Let κ be a double root of the determinant K_{ij} defined in eqn (33) with $q_i = 0$. Then, not only eqns (31) and (32) are the solutions, but also are

$$u_i = v_i\frac{d}{d\kappa}\left\{\frac{A}{1-\kappa}Z^{1-\kappa}\right\} + \bar{v}_i\frac{d}{d\kappa}\left\{\frac{B}{1-\kappa}\bar{Z}^{1-\kappa}\right\} + \cdots \quad (63)$$
$$\sigma_{ij} = \tau_{ij}\frac{d}{d\kappa}\{AZ^{-\kappa}\} + \bar{\tau}_{ij}\frac{d}{d\kappa}\{B\bar{Z}^{1-\kappa}\} + \cdots \quad (64)$$

Since

$$\tau_{ij}\frac{\mathrm{d}}{\mathrm{d}\kappa}\{AZ^{-\kappa}\}=\frac{\mathrm{d}A}{\mathrm{d}\kappa}\tau_{ij}Z^{-\kappa}-A\tau_{ij}Z^{-\kappa}\ln Z, \tag{65}$$

the first term on the right is essentially the same as the first term of eqn (32). The second term provides a new solution for σ_{ij} when κ is a double root:

$$\sigma_{ij}=A_2\tau_{ij}Z^{-\kappa}\ln Z+B_2\bar{\tau}_{ij}\bar{Z}^{-\kappa}\ln\bar{Z}. \tag{66}$$

The real or imaginary parts of eqn (66) have different expressions depending on whether p is a single root or a multiple root.

(a) *p is a single root*

When p is a single root, the real or imaginary parts of eqn (66) have the expression:

$$\begin{aligned}\sigma_{ij}=(rp)^{-\xi}t_{ij}\{&e^{\psi\eta}[M_2^-(\ln(rp)\cos\phi_{ij}^- -\psi\sin\phi_{ij}^-)\\&+N_2^-(\ln(rp)\sin\phi_{ij}^- +\psi\cos\phi_{ij}^-)]\\&+e^{-\psi\eta}[M_2^+(\ln(rp)\cos\phi_{ij}^+ -\psi\sin\phi_{ij}^+)\\&+N_2^+(\ln(rp)\sin\phi_{ij}^+ +\psi\cos\phi_{ij}^+)]\}.\end{aligned} \tag{67}$$

As before, $\phi_{ij}^\mp$ are defined in eqn (39) and $M_2^\pm$, $N_2^\pm$ are related to A_2 and B_2.

(b) *p is a multiple root*

Let us consider first the case in which p is a double root. Then, in addition to eqn (66), we also have the solution

$$\sigma_{ij}=A_2\frac{\mathrm{d}}{\mathrm{d}p}(\tau_{ij}Z^{-\kappa}\ln Z)+B_2\frac{\mathrm{d}}{\mathrm{d}\bar{p}}(\bar{\tau}_{ij}\bar{Z}^{-\kappa}\ln\bar{Z}). \tag{68}$$

However, since

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}p}(\tau_{ij}Z^{-\kappa}\ln Z)=(\tau_{ij}'-\frac{\kappa}{2\beta i}\tau_{ij})Z^{-\kappa}\ln Z\\&+\frac{1}{2\beta i}\tau_{ij}(Z^{-\kappa}-\bar{Z}Z^{-\kappa-1})+\frac{\kappa}{2\beta i}\tau_{ij}\bar{Z}Z^{-\kappa-1}\ln Z\end{aligned} \tag{69}$$

where use has been made of eqn (44), only the last term provides a new solution. The rest of the terms in eqn (69) have appeared in eqns (66), (32) and (46). Therefore, a new solution when p is a double root is

$$\sigma_{ij}=A_2\frac{\kappa}{2\beta i}\tau_{ij}\bar{Z}Z^{-\kappa-1}\ln Z-B_2\frac{\kappa}{2\beta i}\bar{\tau}_{ij}Z\bar{Z}^{-\kappa-1}\ln\bar{Z}. \tag{70}$$

The real or imaginary parts of eqn (70) have the expression which is obtained from eqn (67) with $\phi_{ij}^\mp$ replaced by $(\phi_{ij}^\mp-2\psi)$.

Similarly, if p is a triple root, it is not difficult to show that the new solution is obtained from eqn (67) by replacing $\phi_{ij}^\mp$ by $(\phi_{ij}^\mp-4\psi)$.

We see from eqn (67) that σ_{ij} has the singularity of $r^{-\xi}\ln r$. The existence of a solution of eqn (67) depends on the existence of a solution for A and $\mathrm{d}A/\mathrm{d}\kappa$ in eqn (65). Since A is an element of c_j in eqn (33), the existence of A and $\mathrm{d}A/\mathrm{d}\kappa$ depends on the existence of a solution for c_j and $\mathrm{d}c_j/\mathrm{d}\kappa$ from the following equations

$$K_{ij}c_j=0 \tag{71}$$

$$K_{ij}(\mathrm{d}c_j/\mathrm{d}\kappa)+(\mathrm{d}K_{ij}/\mathrm{d}\kappa)c_j=0. \tag{72}$$

A discussion of the solution of eqns (71) and (72) can be found in[3].

11. STRESS NEAR THE VERTEX WHEN κ IS A TRIPLE ROOT

When κ is a triple root, one can follow the same reasoning as in the previous section for a double root κ. Therefore, the new solution for a triple root κ is obtained by replacing $d/d\kappa$ by $d^2/d\kappa^2$ in eqns (63) and (64). Equation (66) then is replaced by

$$\sigma_{ij} = A_2\tau_{ij}Z^{-\kappa}(\ln Z)^2 + B_2\bar{\tau}_{ij}\bar{Z}^{-\kappa}(\ln \bar{Z})^2 \tag{73}$$

and eqn (67) is modified by replacing $\ln(r\rho)$ by $[\ln(r\rho)]^2 - \psi^2$, which is the real part of $(\ln Z)^2$ and ψ by $2\psi\ln(r\rho)$, which is the imaginary part of $(\ln Z)^2$.

12. GENERAL EXPRESSION

We can summarize the results obtained so far in the following form. Let n_p be the multiplicity of p and m_κ be the multiplicity of κ. If we write

$$\sigma_{ij} = r^{-\xi}F_{ij}(r, \theta) \tag{74a}$$

then F_{ij} consists of a linear combination of the real and imaginary parts of the following expression

$$\begin{aligned} t_{ij}\rho^{-\xi}\, e^{\pm\psi\eta}\{\ln(r\rho) \pm i\psi\}^{m-1}\{&\cos[b_{ij} - \xi\psi - 2(n-1)\psi \mp \eta\ln(r\rho)] \\ &+ i\sin[b_{ij} - \xi\psi - 2(n-1)\psi \mp \eta\ln(r\rho)]\} \end{aligned} \tag{74b}$$

for each p and for all integers m and n subjected to the limitations

$$1 \le m \le m_\kappa, \quad 1 \le n \le n_p \le 3. \tag{74c}$$

As we stated before, ρ and ψ depend on θ but not on r.

13. SINGULARITY AT A CRACK TIP FOR ANISOTROPIC SOLIDS

Consider an infinite anisotropic solid with a crack plane which is located at $x_1 < 0$ of the (x_1, x_3) plane. Hence, $\sigma_{2j} = 0$, ($j = 1, 2, 3$) at $\theta = \pm\pi$. Using eqn (27a), eqn (36) for $\theta = \pi$ and $-\pi$ reduces to

$$\left.\begin{aligned} t_{2j}\{M_1\cos(b_{2j} - \xi\pi) + N_1\sin(b_{2j} - \xi\pi)\} + \ldots = 0 \\ t_{2j}\{M_1\cos(b_{2j} + \xi\pi) + N_1\sin(b_{2j} + \xi\pi)\} + \ldots = 0 \\ (j = 1,2,3). \end{aligned}\right\} \tag{75}$$

If we set $\xi = 1/2$, we have

$$\left.\begin{aligned} t_{2j}\{M_1\sin(b_{2j}) - N_1\cos(b_{2j})\} + \cdots = 0 \\ t_{2j}\{-M_1\sin(b_{2j}) + N_1\cos(b_{2j})\} + \cdots = 0 \\ (j = 1, 2, 3). \end{aligned}\right\} \tag{76}$$

Equation (76) consists of 6 equations for $M_1, N_1, \ldots$ and can be written in the form of eqn (33) with $q_i = 0$. Since each of the three pairs of equations in eqn (76) are identical, $\xi = 1/2$ is at least a triple root of the determinant K_{ij}. We can therefore let $\xi = 1/2$, $\eta = 0$, $m_\kappa = 3$ in eqns (74). Disregarding the dependence on θ, the singularities at the crack tip in a general anisotropic material are $r^{-1/2}$ and possibly $r^{-1/2}\ln r$ and $r^{-1/2}(\ln r)^2$. The existence of $r^{-1/2}\ln r$ and $r^{-1/2}(\ln r)^2$ depends on the existence of a solution for c_j, $dc_j/d\kappa$, $d^2c_j/d\kappa^2$ from eqns (71) and (72) and an equation obtained by differentiating eqn (72) with κ.

For isotropic materials, u_3 is uncoupled from u_1 and u_2 and hence $\xi = 1/2$ is a double root of the determinant K_{ij}. The singularities at the crack tip are $r^{-1/2}$ and possibly $r^{-1/2}\ln r$ if a solution for c_j and $dc_j/d\kappa$ exists. It can be shown that no solutions for c_j and $dc_j/d\kappa$ exist and therefore the singularity $r^{-1/2} \ln r$ is not present at the crack tip of isotropic materials even though $\xi = 1/2$ is a double root.

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